UDC 539.3

ON THE DENSITY OF THE VIBRATION FREQUENCIES OF THIN SHELLS OF REVOLUTION

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Thin shells of revolution, closed in the circumferential direction, and with an arbitrary meridian shape, are considered. The density of the vibration frequencies is determined by using an asymptotic method of integration [1 – 3]. The density of the frequencies in the neighborhoods of condensation points is investigated. The question of the density of vibration frequencies of thin shells has been examined in [4 – 7]. Shallow shells of rectangular planform were examined in [4, 5].

1. Initial equations and their integrals. After insertion of inertial terms and separation of variables the system of three shell equilibrium equations in displacements [8] becomes

$$L_{11}u + L_{12}v + L_{13}w + \lambda u = 0$$

$$L_{21}u + L_{22}v + L_{23}w + \lambda v = 0 \qquad \left(\lambda = \frac{(1-\sigma^2)\gamma\omega^2}{E}\right) \tag{1.1}$$

$$L_{31}u + L_{32}v + L_{33}w + \frac{h^2}{12} N_{33}w + \lambda w = 0$$

Here u, v, w are the projections of the displacements in the direction of the generator, the parallel, and the normal. The linear differential operators L_{ij}, N_{33} depend on the variable coefficients of the quadratic forms of the middle surface and on the number m of waves along a parallel, E is Young's modulus, σ Poisson's ratio, γ the density and ω the vibration frequency.

Let us select the characteristic dimension of the middle surface as the unit of length. Then the shell thickness h will be a small number. Let us introduce the small parameter μ by means of the formula $\mu^4 = h^2 / 12$

Vibration modes with $m \gg 1$ introduce the main contribution to the magnitude of the density. Let us set $m = \mu^{-1}\rho$, then all eight integrals of the system (1.1) have a large index of variability (equal to $^1/_2$) and can be represented as

$$w_k(s, \mu) = \sum_{n=0}^{\infty} \mu^n c_{nk}(s) \exp\left\{\mu^{-1}\int_{s_k}^{s} q_k(t)\,dt\right\}$$

$$c_{0k}(s) = \left(q_k^2 - \frac{\rho^2}{B^2}\right)\left(B\frac{\partial f}{\partial q_k}\right)^{-1/2} \qquad (k = 1, 2, \ldots, 8) \tag{1.2}$$

$$f(q) \equiv \left(q^2 - \frac{\rho^2}{B^2}\right)^4 - \lambda\left(q^2 - \frac{\rho^2}{B^2}\right)^2 + (1-\sigma^2)\left(\frac{q^2}{R_2} - \frac{\rho^2}{B^2R_1}\right)^2 = 0 \tag{1.3}$$

where q_k are the roots of the characteristic equation, s is the arclength of a generator, $B(s)$ the distance to the axis of rotation, $R_1(s)$, $R_2(s)$ the principal radii of curvature (R_1 is the radius of curvature of the meridian). Let us consider $R_2 > 0$, and let us impose no constraint on the sign of R_1. The other unknown quantities (the displacements u, v, the stress resultants T_1, T_2, S, N_1, N_2, the moments M_1, M_2, H) are also represented in series of the form (1.2).

The solutions (1.2) are applicable in the neighborhoods of those points s in which the roots of (1.3) are pairwise distinct. The points $s = s_*$ at which $q_k(s) = q_l(s)$ are called turning points. The solutions (1.2) are unsuitable in the neighborhoods of these points because $c_{0k}(s_*) = \infty$. Solutions of the system (1.1) in the neighborhoods of turning points have been constructed in [2, 3] and formulas have been found connecting solutions of the form (1.2) which have passed through a turning point. The results in [2, 3] are used below.

2. On the density of frequencies for fixed m. Let us consider a shell bounded by the parallels $s = s_1$ and $s = s_2$, on which four homogeneous boundary conditions have been given (in the case of a dome we demand boundedness of the solutions at the vertex). The eighth order determinant obtained by substituting a linear combination of solutions in the boundary conditions set equal to zero will be the frequency equation. Equation (1.3) can have a different number of pure imaginary roots depending on the parameters m, s, ω. Setting aside the case of multiple roots, we obtain three cases, provisionally denoted by A_i $(i = 0, 1, 2)$, where i is the number of pairs of imaginary roots. The case A_1 holds for

$$\lambda > \lambda_0 = \frac{m^4 h^2}{12 B^4} + \frac{1-\sigma^2}{R_1^2} \tag{2.1}$$

and the cases A_0 or A_2 for $\lambda < \lambda_0$. If

$$\frac{m^4 h^2}{12 B^4} > \frac{1-\sigma^2}{R_1}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{2.2}$$

then the case A_0 holds for all $\lambda < \lambda_0$. Otherwise for $\lambda_* < \lambda < \lambda_0$ case A_2 holds, and for $\lambda < \lambda_*$ case A_0. Here

$$\lambda_* = \min_{\varkappa \geqslant 0}\left\{\frac{\rho^4(\varkappa+1)^2}{B^4} + \frac{1-\sigma^2}{(\varkappa+1)^2}\left(\frac{\varkappa}{R_2} + \frac{1}{R_1}\right)\right\} \tag{2.3}$$

Let $m \gg 1$ be fixed. It has been shown in [1 – 3] that if the case A_0 or the case A_1 holds for all s, then the frequency equation is

$$\operatorname{tg}\left\{\int_{D_1} \mu^{-1} |q_1(s, \omega)|\, ds + \varphi(\omega)\right\} = \Psi(\omega) \tag{2.4}$$

where q_1 is the pure imaginary root, $D_1(m, \omega)$ is the part of the segment (s_1, s_2) on which it exists. In connection with the fact that the functions $\varphi(\omega)$ and $\Psi(\omega)$ vary slowly, the density $n(m, \omega)$ of the roots of Eq. (2.4) is

$$n(m, \omega) = \frac{1}{\pi\mu} \int_{D_1} \left|\frac{\partial q_1}{\partial \omega}\right| ds \tag{2.5}$$

Formula (2.5) has been derived for rigidly fixed edges, however it also holds for other boundary conditions. Only the functions φ and Ψ depend on their form.

If (1.3) has two pairs of pure imaginary roots $\pm q_1$ and $\pm q_2$ (let $|q_1| > |q_2|$), then for arbitrary boundary conditions the density $n(m, \omega)$ cannot be found successfully. In the case of the Navier boundary conditions ($T_1 = v = w = M_1 = 0$ for $s = s_1$ $s = s_2$) the frequency equation decomposes into two equations

$$\sin\left\{\int\limits_{D_k} \mu^{-1} \,|\, q_k(s, \omega) \,|\, ds\right\} + O(\mu) = 0 \qquad (k = 1, 2) \tag{2.6}$$

where, as before, D_k is the domain of existence of the imaginary root q_k. It follows from (2.6) that

$$n(m, \omega) = \frac{1}{\pi\mu}\left(\int\limits_{D_1}\left|\frac{\partial q_1}{\partial\omega}\right| ds + \int\limits_{D_2}\left|\frac{\partial q_2}{\partial\omega}\right| ds\right) \tag{2.7}$$

Formulas (2.7) also holds for rigidly fixed shell edges $(u = v = w = w' = 0)$ if $m = O(\mu^{-1/2})$. Indeed if

$$R_1 R_2 < 0, \qquad \lambda < (1 - \sigma^2)\min_s\{R_1^{-2}, R_2^{-2}\}$$

the frequency equation reduces to [1]

$$a_1 \cos\left[\mu^{-1}\int\limits_{s_1}^{s_2}(|q_2| - |q_1|)\, ds\right] = 4 + a_2\cos\left[\mu^{-1}\int\limits_{s_1}^{s_2}(|q_1| + |q_2|)\, ds\right]$$

$$q_{1,2} = \frac{\mu m}{B}\left(\sqrt{\frac{\lambda}{1-\sigma^2}} \pm \frac{1}{R_1}\right)^{1/2}\left(\sqrt{\frac{\lambda}{1-\sigma^2}} \pm \frac{1}{R_2}\right)^{-1/2} \tag{2.8}$$

Formula (2.7) holds because

$$a_1 > 4 + a_2, \qquad \frac{\partial}{\partial\omega}|q_1| < 0, \ \frac{\partial}{\partial\omega}|q_2| > 0$$

Let us examine the case of small $m = O(1)$ (including $m = 0$). The boundary conditions are arbitrary. Let us set

$$b(s, \omega) = \lambda - (1 - \sigma^2) R_2^{-2}, \qquad \lambda^- = \min_s\left\{\frac{1-\sigma^2}{R_2^2}\right\}, \qquad \lambda^+ = \max_s\left\{\frac{1-\sigma^2}{R_2^2}\right\}$$

Then for $\lambda > \lambda^+$ the density is [1]

$$n(m, \omega) = \frac{1}{\pi\mu}\frac{\partial}{\partial\omega}\int\limits_{s_1}^{s_2} b^{1/4}\, ds \tag{2.9}$$

and upon compliance with the conditions $\lambda^- < \lambda < \lambda^+$, $R_2'(s) \neq 0$ $(s_1 \leqslant s \leqslant s_2)$ the density is determined by (2.9) in which the integration must be carried out over the part of the interval (s_1, s_2) wherein $b \geqslant 0$. It follows from (2.9) that for $\lambda > \lambda^-$ the density increases as the thickness h diminishes. The number of frequencies is practically independent of h for $\lambda < \lambda^- - \varepsilon$, hence we consider that $n(m, \omega) \approx 0$ for $\lambda < \lambda^-$

3. Frequency density in the general case. The density $n(\omega)$ is

$$n(\omega) = n(0, \omega) + 2\sum_{m\neq 0} n(m, \omega) \tag{3.1}$$

(the densities $n(m, \omega)$ are found in Sect. 2). The factor two in (3.1) appears because two vibration modes ($w(s)\cos m\varphi$ and $w(s)\sin m\varphi$, where φ is the longitudinal angle) correspond to each root of the frequency equation for $m \neq 0$

Let us consider approximately that the density $n(m, \omega)$ is determined by (2.7) for all m (we consider (2.5) as a particular case of (2.7)). Hence, an error will be committed for the components with $n = O(1)$, whose order will be estimated below. We obtain

$$n(\omega) = \frac{2}{\pi\mu} \sum_m \left(\int_{D_1} \left| \frac{\partial q_1}{\partial \omega} \right| ds + \int_{D_2} \left| \frac{\partial q_2}{\partial \omega} \right| ds \right) \tag{3.2}$$

Let us introduce the notation

$$q = i\frac{\mu m r}{B}, \qquad D = \frac{Eh^3}{12(1-\sigma^2)}, \qquad \omega_k(s) = \left(\frac{E}{\gamma}\right)^{1/2} \frac{1}{R_k(s)} \ (k = 1, 2), \qquad \chi = R_2 R_1^{-1}$$

($\omega_1 < 0$ for $R_1 < 0$). Then (1.3) can be rewritten as

$$\Phi(r, s, m, \omega) = \frac{Dm^4}{\gamma h B^4}(r^2+1)^4 - \omega^2(r^2+1)^2 + (\omega_2 r^2 + \omega_1)^2 = 0 \tag{3.4}$$

In (3.2) we replace approximately the summation over m by integration

$$n(\omega) = \frac{2}{\pi} \int_{s_1}^{s_2} \frac{\partial}{\partial \omega} \left(\int_0^{m_1} \frac{m r_1}{B} dm - \int_{m_2}^{m_1} \frac{m r_2}{B} dm \right) ds \tag{3.5}$$

where $r_k(s, m, \omega)$ are the positive roots of (3.4) $(r_1 \geqslant r_2)$. The limits of integration define the domains (perhaps empty) of existence of these roots. Since $dr_2 / \partial\omega < 0$, there is hence a minus in (3.5). Using the identity

$$\frac{\partial r}{\partial \omega} = \frac{\partial \Phi}{\partial \omega} \left(\frac{\partial \Phi}{\partial m} \right)^{-1} \frac{\partial r}{\partial m} \tag{3.6}$$

we reduce (3.5) to

$$n(\omega) = \frac{1}{\pi} \int_{s_1}^{s_2} B(\gamma h D^{-1})^{1/2} \operatorname{Re}\left\{ \int_0^\infty [\omega^2(r^2+1)^2 - (\omega_2 r^2 + \omega_1)^2]^{-1/2} \omega \, dr \right\} ds \tag{3.7}$$

where $\operatorname{Re}\{z\}$ is the real part of the number z. From (3.7) we obtain

$$n(\omega) = \frac{1}{2} \int_{s_1}^{s_2} B(\gamma h D^{-1})^{1/2} G(\omega, s)\, ds \tag{3.8}$$

$$G = \begin{cases} \dfrac{2}{\pi} H(\omega, s) F[k(\omega, s)] & \text{for} \quad R_1(s) \neq R_2(s) \\ (1 - a_1^{-2})^{-1/2} & \text{for} \quad R_1(s) = R_2(s) \end{cases} \tag{3.9}$$

$$H = 0 \quad \text{for} \quad 0 < a_1, \quad a_2 < 1$$

$$H = H_1 = \left[\frac{a_1 a_2}{2(a_2 - a_1)} \right]^{1/2}, \quad k = k_1 = \left[\frac{(a_2-1)(a_1+1)}{2(a_2-a_1)} \right]^{1/2}$$
$$\text{for} \quad 0 < a_1 < 1, \ a_2 > 1$$

$$H = H_2, \quad k = k_2 \quad \text{for} \ |a_1| > 1, \quad a_2 < 1 \quad \text{or} \quad -1 < a_1 < 0, \quad a_2 > 1 \tag{3.10}$$

$$H = H_1' = \left[\frac{a_1 a_2}{(a_1+1)(a_2-1)} \right]^{1/2}, \quad k = k_1^{-1} \quad \text{for} \ 1 < a_1 < a_2$$

$$H = H_2', \quad k = k_2^{-1} \quad \text{for} \ 1 < a_2 < a_1 \quad \text{or} \quad a_1 < -1, \quad a_2 > 1$$
$$\text{or} \quad -1 < a_1 < 0, \quad a_2 < 1$$

Here $F(k)$ is the complete elliptical integral of the first kind, $a_1 = \omega / \omega_1$, $a_2 =$

$= \omega / \omega_2 > 0$. The quantities k_2, H_2, H_2' are obtained from k_1, H_1, H_1' by commutating the subscripts 1 and 2. Graphs of the function G are shown in Fig. 1 for shells of positive and negative Gaussian curvature. If the shell parameters B, R_1, R_2 are independent of s (almost cylindrical shells, for example), then results (3.8) – (3.10) differ from the results in [4, 5] only by notation. Presented there are also graphs corresponding to Fig. 1. Formula (3.8) can be integrated thus for an arbitrary shell of revolution. Let us divide the shell into small rectangles with constant parameters B, R , R_2, let us find the frequency density for each of them by the formula in [4, 5], and let us then sum these densities. Such a method of calculating the density was proposed in [6] as a hypothesis. Cases when (3.8) has a foundation are discussed below.

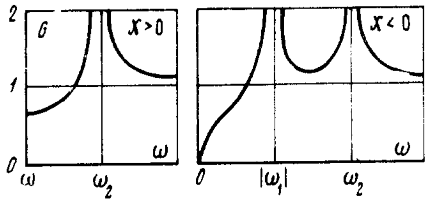

Fig. 1.

For $\omega \to \infty$ formula (3.8) goes over into the Courant formula [8] for the density of the transverse vibration frequencies of a plate of area S

$$n_0 = \frac{S}{4\pi}\left(\frac{\gamma h}{D}\right)^{1/2} \tag{3.11}$$

Indeed for $\omega \to \infty$ we have $G \to 1$, and (3.11) is obtained if it is taken into account that the shell area is

$$S = 2\pi \int_{s_1}^{s_2} B\, ds \tag{3.12}$$

Passage to the limit as $\omega \to \infty$ is provisional because the initial equations (1.1) are inapplicable at high frequencies. However, for sufficiently thin shells a range of frequencies exists in which the equations (1.1) are applicable, and the density $n(\omega)$ is close to n_0 at the same time.

Terms of the form (2.9) were discarded in going from (3.1) to (3.2). The order of the error admitted here is determined by the ratio between the densities (2.9) and (3.11)

$$n(m, \omega) / n_0 = O(\mu) = O[(h / R)^{1/2}] \tag{3.13}$$

where R is the characteristic dimension of the middle surface.

It can be assumed that (3.8) holds to any degree of accuracy in all cases independently of the frequency ω, the kind of boundary conditions, and the shape of the middle surface. However, (3.8) does not always have a foundation

Formula (3.8) has been obtained from (2.7), which is proved for all m in the following cases:

1) For any ω and boundary conditions if $0 \leqslant \chi(s) < 1$;

2) For any boundary conditions and shells of any shape if there is compliance with one of the inequalities

$$\omega < \omega_- = \min_s \{\omega_1, \omega_2\}, \qquad \omega > \omega_+ = \max_s \{\mathrm{Re}\,(\omega_2 \sqrt{2\chi^2 - \chi})\}$$

3) For any ω and shells of any shape if the Navier boundary conditions hold.

In cases (1) and (2) Eq. (1.3) has not more than one pair of pure imaginary roots for all s and (2.5) is used to calculate the density. In all the remaining cases (1.3) has two pairs of pure imaginary roots for some m, s and (2.7) has a foundation only for the

Navier boundary conditions. As in [10], let us say that in these cases degeneration of the edge effect holds. This latter circumstance makes giving a foundation for the formula for the density difficult for $\omega < \omega_+$ for both shells of revolution and for shallow shells of rectangular planform [6]. If B, R_1, R_2 are constants, the inequality $\omega < \omega_+$ differs from the inequality (8) in [6] only in notation.

Let us discuss (3.8) from another viewpoint. It has been noted in [4, 5] that $G(s, \omega)$ becomes infinite for $\omega = \max\{\omega_1, \omega_2\}$ in the case $\chi \geqslant 0$ and for $\omega = -\omega_1$, $\omega = \omega_2$ in the case $\chi < 0$. For shells of revolution $G(s, \omega)$ is obtained when evaluating the inner integral in (3.7). It is improper and diverges for the above-mentioned values of ω either for $r = 0$ (in the case $\omega = |\omega_1|$), or for $r = \infty$ in the case $\omega = \omega_2$). From (3.3) we have $q / \mu = imrB^{-1}$. Therefore, for $r = 0$ the integral (1.2) is not an integral with a high index of variability, and the method of constructing the frequency equation becomes inaccurate near $r = 0$.

For $r = \infty$ we have $m = 0$; it was assumed in deriving (3.8) that the members with $m = O(\mu^{-1})$ introduce the main contribution, and this is not so in this case.

Thus, the infinite discontinuities in $G(s, \omega)$ are a consequence of imperfections in the method in both cases. When $R_1(s)$ and $R_2(s)$ are not constants, the discontinuities vanish after integration in (3.8), and this formula should apparently yield satisfactory results.

Cases when (3.8) yields infinite discontinuities and expressions more exact than (3.8) are obtained for the densities are examined below. Hence modes with $m = O(\mu^{-3/2})$ yield the main contribution in the neighborhoods of frequency concentration points. The contribution of the mode with $m = O(1)$ is small because of (3.13).

The case $m = O(\mu^{-t})$ $(1 < t < 2)$ can be obtained from the case $m = \mu^{-1}\rho$ considered for $\rho \gg 1$ and corresponds to frequencies for which the density is close to the Courant density.

4. Spherical shell. For a spherical shell of radius R_1 formula (3.8) yields

$$\frac{n(\omega)}{n_0} = \begin{cases} 0, & \omega < \omega_1 = (E/\gamma)^{1/2} R_1^{-1} \\ [1 - (\omega_1/\omega)^2]^{-1/2}, & \omega > \omega_1 \end{cases} \tag{4.1}$$

For a shallow shell of rectangular planform (4.1) has been obtained in [4, 5]. A shell bounded by two parallels, in the shape of a dome or a closed shell, is examined below in the neighborhood of a condensation point of the frequency $\omega = \omega_1$.

A formula has been obtained in [11] to determine frequencies of the form (2.4), from which we find

$$n(m, \omega) = \frac{1}{\pi}\frac{\partial}{\partial\omega}\int\limits_{D_1}\left(p_1 - \frac{m^2}{\sin^2\theta}\right)^{1/2} d\theta \tag{4.2}$$

where θ is the angle of latitude on a sphere $(0 \leqslant \theta \leqslant \pi)$, D_1 is the part of the interval $\theta_1 \leqslant \theta \leqslant \theta_2$ in which the radicand is nonnegative, and p_1 is a positive root of the equation

$$\mu_1^4 p^3 + (1 - \sigma^2 - \lambda_1)(p - 2 - \lambda_1) - (1 + \sigma)(2 + \sigma)\lambda_1 = 0$$

$$(\lambda_1 = \lambda R_1^2,\ \mu_1^4 = h^2 / (12 R_1^2)) \tag{4.3}$$

As in Sect. 3, summing over m we find

$$n(\omega) = \frac{S}{4\pi R_1^2}\frac{dp_1}{d\omega}, \quad S = 2\pi R_1^2(\cos\theta_1 - \cos\theta_2) \tag{4.4}$$

The same formula can be obtained for a closed sphere also by a direct computation of the number of solutions corresponding to frequencies less than ω which are bounded at the poles.

An approximate representation of $p_1(\omega)$ for $\omega \approx \omega_1$ and $\omega > \omega_1$ is given by the parametric formulas [11]

$$p_1 = \mu_1^{-4/3} y, \quad \omega^2 = \omega_1^2 \left[1 + \frac{\mu_1^{4/3}(y^3 - c)}{(1-\sigma^2)\, y} \right] \tag{4.5}$$
$$(c = (1+\sigma)(2+\sigma)(1-\sigma^2))$$

where the parameter y varies between $0 < y < \infty$. Then (4.4) yields

$$\frac{n(\omega)}{n_0} = f(y) = \frac{y^2}{y^3 + c/2} [(1 - \sigma^2)\mu_1^{-4/3} + (y^3 - c)\, y^{-1}]^{1/3} \tag{4.6}$$

For $\omega = \omega_1$ the coefficient of concentration $f(y)$ is a maximum

$$f_+ = \max\{f(y)\} = (12 R_1^2 / h^2)^{1/6}\, {}^2/_3 (1 - \sigma^2)^{1/2} c^{-1/3} \tag{4.7}$$

and for $\omega \to \infty$ we have $y \to \infty$ and $f(y) \to 1$, i.e., the Courant density is obtained in the limit. We have $f_+ = 3.2$ for $R_1 / h = 100$, $\sigma = 0.3$, and the function $f(y)$ is shown in Fig. 2 by the solid line (the dashes are (4.1)).

5. Cylindrical shell. The density has an infinite discontinuity of the form $n(\omega) = O(\ln|\omega - \omega_2|)$ [4, 5] for a cylindrical shell of radius R_2 at $\omega = \omega_2$. The same result also follows from (3.8). The density in the neighborhood of $\omega = \omega_2$ has been investigated in more detail in [7], where it has been shown that the maximum coefficient of concentration is on the order of

$$n(\omega) / n_0 = O[\ln(R_2 / h)] \tag{5.1}$$

An "experimental" construction of the density $n(\omega)$ for a cylindrical and spherical shell was carried out in [5]. The results obtained are in qualitative agreement with (5.1) and (4.7).

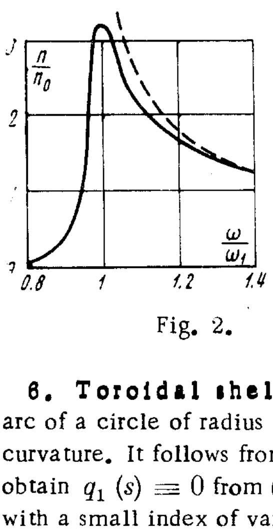

Fig. 2.

6. Toroidal shell. Let us consider the part of a torus formed by rotating the arc of a circle of radius $-R_1$ $(R_1 < 0)$ around an axis and having negative Gaussian curvature. It follows from (3.8) that the density is $n(\omega) = \infty$ for $\omega = -\omega_1$. We obtain $q_1(s) \equiv 0$ from (2.8) for $\omega = -\omega_1$ which indicates the existence of integrals with a small index of variability in the system (1.1). Let us construct them. Let

$$R_2 < -R_1, \quad 1 \ll m \ll \mu^{-1}, \quad \lambda = (1 - \sigma^2) / R_1^2 + \lambda_2 / m^2 \tag{6.1}$$

Then two solutions of the system (1.1) can be represented as power series in m^{-1}

$$\begin{gathered} u(s) = \frac{1}{m^2}\left(B\frac{d}{ds}\left(\left(\frac{1}{R_1} + \frac{\sigma}{R_2}\right)\frac{y}{B}\right) - \frac{2(1+\sigma)}{R_1}\frac{dy}{ds}\right) + O\left(\frac{1}{m^3}\right) \\ v(s) = \frac{1}{m}\left(\frac{1}{R_2} + \frac{\sigma}{R_1}\right) y + O\left(\frac{1}{m^2}\right), \quad w(s) = \frac{y}{B} + O\left(\frac{1}{m}\right) \end{gathered} \tag{6.2}$$

where $y(s)$ satisfies the equation

$$y'' + g(s, \omega)\, y = 0, \qquad g = m^6 h^2 g_1 - m^2 (\lambda - (1 - \sigma^2) R_1^{-2})\, g_2 - g_3$$

$$g_1(s) = \frac{R_1^2 R_2}{24(1-\sigma^2) B^6 (R_2 - R_1)}, \qquad g_2(s) = 12 B^4 g_1(s) \tag{6.3}$$

$$g_3(s) = \frac{3R_1^2 + (3\sigma - 1) R_1 R_2 + \sigma^2 R_2^2}{2 R_1^2 R_2 (R_2 - R_1)} - \left(\frac{B'}{B}\right)^2$$

Let us consider $m = O(h^{-1/4})$, then $g = O(1)$. The remaining six integrals of (1.1) have a high index of variability [1]. Two have the index $^1/_3$ and oscillate, and four have the index $^1/_2$ and the character of integrals of the edge effect.

In the case of the Navier boundary conditions the frequency equation in a first approximation decomposes into the product of two equations, of which the first results in the boundary value problem

$$y'' + g(s, \omega) y = 0, \qquad y(s_1) = y(s_2) = 0 \tag{6.4}$$

and the second yields

$$\int_{s_1}^{s_2} \frac{m}{B} \left(\frac{\omega - \omega_1}{\omega_2 - \omega}\right)^{1/2} ds = k\pi \qquad (k = 1, 2, \ldots) \tag{6.5}$$

Let us find $n(-\omega_1)$. There is no single formula here suitable for all m. We will use (3.7) for solutions with high index of variability, and (6.4) and (6.5) for a low index. Let us add the results

$$n(\omega) = n_1(\omega) + n_2(\omega) + n_3(\omega) \tag{6.6}$$

Let us use (3.7) for $\omega = -\omega_1$ $(r \gg r_0 > 0)$

$$n_1(\omega) = \frac{1}{\pi} \int_{s_1}^{s_2} \left(\frac{\gamma h B^2 R_2^2}{D(R_1^2 - R_2^2)}\right)^{1/2} \int_{r_0}^{b_2} \frac{dr}{r(b_2^2 - r^2)^{1/2}}\, ds \qquad \left(b_2^2 = -\frac{2R_2}{R_1 + R_2}\right) \tag{6.7}$$

As in Sects. 2 and 3, we find from (6.4)

$$n_2(\omega) = \frac{2}{\pi} \int_{s_1}^{s_2} \int_{m_1}^{m_2} \left| \frac{\partial}{\partial \omega} g^{1/2} \right| dm\, ds \tag{6.8}$$

where we find the limit m_1 from the condition $g \gg 0$, and the limit m_2 from the condition that the range of integration in (6.8) include the domain $r < r_0$ omitted in (6.7)

$$m_1^6 = \frac{g_3}{h^2 g_1}, \qquad g = \left(\frac{m_2 r_0}{B}\right)^2, \qquad m_2^4 = \frac{r_0^2}{B^2 h^2 g_1} + \frac{g_3}{m_2^2 h^2 g_1} \tag{6.9}$$

For $r_0^2 \gg m^{-2}$ the second member in the expression for m_2^4 can be neglected. The number m falls in the domain $m \ll \mu^{-1}$ for $r_0 \ll 1$. For $\omega = -\omega_1$ we find from (6.8)

$$n_2(\omega) = \frac{1}{3\pi} \int_{s_1}^{s_2} \left(\frac{2\gamma h R_2 B^2}{D(R_2 - R_1)}\right)^{1/2} \ln\left[\left(\frac{m_2}{m_1}\right)^3 + \left(\left(\frac{m_2}{m_1}\right)^6 - 1\right)^{1/2}\right] ds \tag{6.10}$$

Finally, we see that for $r_0 < 1$ Eq. (6.5) yields no contribution in the domain $r < r_0$, i.e., $n_3(\omega) = 0$. If $(m_2 / m_1)^3 \gg 1$, the expression (6.10) simplifies and added to (6.7) yields

$$n(-\omega_1) = \frac{1}{4\pi} \int_{s_1}^{s_2} \left(\frac{2\gamma h R_2 B^2}{D(R_2 - R_1)}\right)^{1/2} \ln\left(\frac{16 R_2 2^{1/3}}{B^2 \, | R_1 + R_2 | \, (h^2 g_1 g_3^2)^{1/3}}\right) ds \tag{6.11}$$

Quantities on the order of r_0^2 as compared with one were discarded in obtaining (6.11).

Hence, for $\mu^{2/3} \ll r_0 \ll 1$ the density is independent of r_0, which indicates the existence of a domain of values r in which the integrals (1.2) and (6.2) are simultaneously suitable. In this case the coefficient of concentration of the density has the order of $\ln |R_1 / h|$.

Let us consider an example. Let

$$R_1 = -R,\ R = 100\,h,\ B = R\,(1.3 - \cos\theta),\ |\theta| \leqslant \pi / 6,\ \sigma = 0.3$$

Then (6.11) yield $n / n_0 = 5.0$.

BIBLIOGRAPHY

1. Tovstik P.E., Integrals of a system of equations of nonaxisymmetric vibrations of a thin shell of revolution. Collection of: Investigations on Elasticity and Plasticity, №5, Leningrad Univ. Press, 1966.
2. Tovstik P.E., On the frequency spectrum of vibrations of shells of revolution with a high wave number along the parallel in a singular case. Collection of: Investigations on Elasticity and Plasticity, №5, Leningrad Univ. Press, 1966.
3. Tovstik P.E., On the frequency spectrum of vibrations of shells of revolution with a high wave number along the parallel. Transactions 6-th All-Union Conference on Plate and Shell Theory (Baku, 1966), Moscow, "Nauka", 1967.
4. Bolotin V.V., On the density of distribution of natural frequencies of thin elastic shells. PMM Vol.27, №2, 1963.
5. Bolotin V.V., The density of eigenvalues in vibrations problems of elastic plates and shells. Warsaw, Proc. Vibration Problems, Vol.6, №4, 1965.
6. Gol'denveizer A.L., On the frequency distribution density of oscillations of a thin shell. PMM Vol.34, №5, 1970.
7. Skuchik E., Simple and Complex Vibrational Systems. Moscow, "Mir", 1971.
8. Gol'denveizer A.L., Some mathematical problems of linear thin elastic shell theory. Uspekhi Matem. Nauk, Vol.15, №5, 1960.
9. Courant R. and Hilbert D., Methods of Mathematical Physics (Russian Translation), Vol.1, Moscow-Leningrad, GOSTEKHIZDAT, 1951.
10. Bolotin V.V., The edge effect in the oscillation of elastic shells, PMM Vol. 24, №5, 1960.
11. Tovstik P.E., Free vibrations of a thin spherical dome. Col. Investigations in Elasticity and Plasticity, №6, Leningrad Univ. Press, 1967.

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